MAT102 ANALİZ II FİNAL SINAV SORULARI VE ÇÖZÜMLERİ

1) a) $\ln(a+x) < \ln a + \frac{x}{a}$, $(x \geq -a, x \neq 0, a>0)$ olduğunu göst.

Çözüm:

$f(x) = \ln(x+a) - \ln a - \frac{x}{a}$ fonksiyonunu tanımlayalım.

f, $(-a,\infty)$ aralığında türevlidir ve

$f'(x) = \frac{1}{a+x} - \frac{1}{a} = \frac{-x}{a(a+x)}$ dır. $\Rightarrow$

x	$-a$	0 ∞
f'	$+$	$-$
f	$\nearrow$	$\searrow$

$x \in (-a,0)$ için $f'(x) > 0 \Rightarrow f$ artan

$x \in (0,\infty)$ için $f'(x) < 0 \Rightarrow f$ azalandır.

$f(0) = 0 \quad -a < x < 0 \Rightarrow f(x) < f(0) = 0$

$\ln(x+a) - \ln a - \frac{x}{a} < 0 \Rightarrow \ln(x+a) < \ln a + \frac{x}{a}$ bulunur.

b) $f(x) = \int_{x-\pi}^{\ln(x+1)} \frac{dt}{\cos t + 2}$ olduğuna göre $f'(0) = ?$

Çözüm: $u(x) = x-\pi$, $v(x) = \ln(x+1)$, $f(t) = \frac{1}{\cos t + 2}$ olsun.

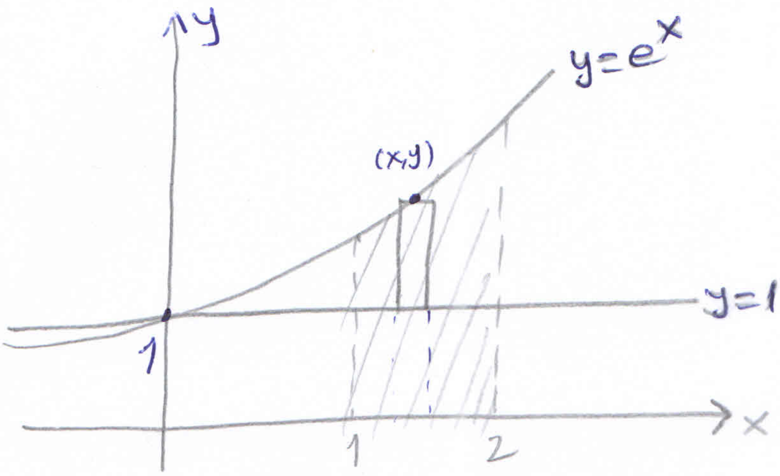
$f'(x) = v'(x) \cdot f(v(x)) - u'(x) \cdot f(u(x))$

$$f'(x) = \frac{1}{x+1} \cdot \frac{1}{\cos(\ln(x+1))+2} - 1 \cdot \frac{1}{\cos(x-\pi)+2}$$

$$f'(0) = \frac{1}{0+1} \cdot \frac{1}{\cos(\ln 1)+2} - \frac{1}{\cos(-\pi)+2} = \frac{1}{1+2} - \frac{1}{1+2} = 0 \Rightarrow \boxed{f'(0)=0}$$

2) a) $y = e^x$ eğrisinin $x=1$, $x=2$ doğruları arasında kalan parçasının $y=1$ doğrusu etrafında dönmesi sonucu oluşan cismin hacmini bulunuz.

Çözüm:



$$V = \pi \cdot \int_1^2 (y-1)^2 \, dx$$

$$= \pi \cdot \int_1^2 (e^x-1)^2 \, dx$$

$$= \pi \cdot \int_1^2 (e^{2x} - 2e^x + 1) \, dx$$

$$= \pi \left[\tfrac{1}{2}e^{2x} - 2e^x + x\right]_1^2$$

$$= \pi \left[\tfrac{1}{2}e^4 - 2e^2 + 2 - \tfrac{1}{2}e^2 - 2e^1 + 1\right]$$

$$= \pi \left[\frac{e^4}{2} - \frac{5}{2}e^2 - 2e + 3\right] \text{ br}^3$$

② b) $I=\int_0^1 \frac{2x^3-4x^2-x-3}{x^2-2x-3}\,dx$ integralini hesaplayınız.

Çözüm:

$$I=\int_0^1\left[2x+\frac{5x+3}{x^2-2x-3}\right]dx=\int_0^1 2x\,dx+\int_0^1\frac{3}{x-3}\,dx+\int_0^1\frac{2}{x+1}\,dx$$

$$=\left[x^2+3\ln|x-3|+2\ln|x+1|\right|_0^1=1+\ln\frac{27}{32}$$

$$\left[\frac{5x+3}{(x-3)(x+1)}=\frac{A}{x-3}+\frac{B}{x+1},\quad \begin{matrix}A+B=5\\ A-2B=-3\end{matrix}\quad A=3,\ B=2\right]$$

③ a) $f(x)=x+\int\sqrt{x^2+a}\,dx$ eşitliği ile verilen $f(x)$ fonksiyonunun $x=1$ absisli noktasındaki teğetinin eğimi 5 ise $a=?$

Çözüm: $f(x)-x=\int\sqrt{x^2+a}\,dx \Rightarrow f'(x)-1=\sqrt{x^2+a} \Rightarrow$

$f'(x)=1+\sqrt{x^2+a}$, $\quad 5=f'(1)=1+\sqrt{1^2+a} \Rightarrow f'(1)=\sqrt{1+a}+1$

$\sqrt{1+a}=4 \Rightarrow \boxed{a=15}$

b) $\lim\limits_{x\to 0}(x+\sin x)^{\tan x}=?$ $\qquad f(x)=(x+\sin x)^{\tan x} \Rightarrow$

$$\ln f(x)=\tan x\ln(x+\sin x)=\frac{\ln(x+\sin x)}{\cot x}$$

$$\lim_{x\to 0}\ln f(x)=\lim_{x\to 0}\frac{\ln(x+\sin x)}{\cot x}\overset{\frac{\infty}{\infty}}{=}\lim_{x\to 0}\frac{\frac{1+\cos x}{x+\sin x}}{-\frac{1}{\sin^2 x}}$$

$$=\lim_{x\to 0}\frac{(1-\sin^2 x)(1+\cos x)}{x+\sin x}=\lim_{x\to 0}\frac{(-\sin x)(1+\cos x)}{\frac{x}{\sin x}+1}=\frac{0}{1+1}=0$$

$$\lim_{x\to 0}(x+\sin x)^{\tan x}=e^0=1$$

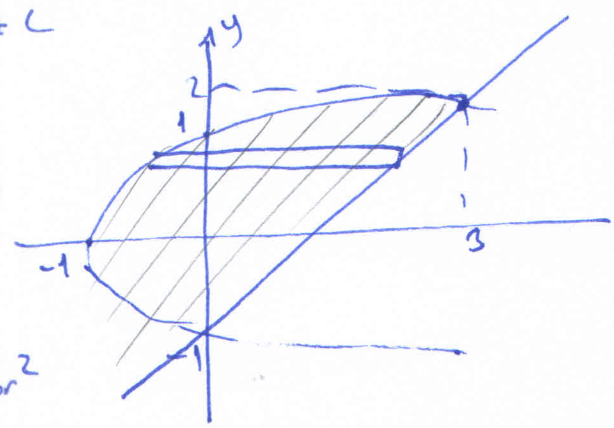
④ a) $I=\int\frac{x^2-1}{(x^4+3x^2+1)\arctan\left(\frac{x^2+1}{x}\right)}=?$

Çözüm:
$u=\arctan\left(\frac{x^2+1}{x}\right) \Rightarrow$

$du=\frac{x^2-1}{x^4+3x^2+1}\,dx \Rightarrow$

$$I=\int\frac{du}{u}=\ln|u|+c=\ln\left|\arctan\left(\frac{x^2+1}{x}\right)\right|+c$$

b) Çözüm: $\left.\begin{matrix}y^2=x+1\\ y=x-1\end{matrix}\right\} \Rightarrow y_1=-1,\ y_2=2$
$\qquad x_1=0,\ x_2=3$



$$\int_{-1}^{2}(y+1-y^2+1)\,dy=\int_{-1}^{2}(2+y-y^2)\,dy$$

$$=\left(2y+\frac{y^2}{2}-\frac{y^3}{3}\right|_{-1}^{2}=4\tfrac{1}{2}\ \text{br}^2$$

5a) $\displaystyle\int_{-1}^{2} |x|.[\![x^2]\!]\,dx$ integrali için kritik noktalar:

$$|x| \Rightarrow x=0$$

$$[\![x^2]\!] \Rightarrow x^2\in\mathbb{Z} \Rightarrow x=\mp1,\mp\sqrt{2},\mp\sqrt{3},\ldots$$

$$\int_{-1}^{2}|x|.[\![x^2]\!]dx=\int_{-1}^{0}|x|.[\![x^2]\!]dx+\int_{0}^{1}|x|.[\![x^2]\!]dx$$

$$+\int_{1}^{\sqrt{2}}|x|.[\![x^2]\!]dx+\int_{\sqrt{2}}^{\sqrt{3}}|x|.[\![x^2]\!]dx$$

$$+\int_{\sqrt{3}}^{2}|x|.[\![x^2]\!]dx$$

$$=0+0+\int_{1}^{\sqrt{2}}x.dx+\int_{\sqrt{2}}^{\sqrt{3}}2x.dx+\int_{\sqrt{3}}^{2}3x.dx$$

$$=\frac{x^2}{2}\Big|_{1}^{\sqrt{2}}+x^2\Big|_{\sqrt{2}}^{\sqrt{3}}+\frac{3x^2}{2}\Big|_{\sqrt{3}}^{2}=\frac{1}{2}+1+\frac{3}{2}=3.$$

5b)

$$J=\int\frac{x+3}{\sqrt{x^2+4x-4}}dx=\int\frac{2x+6}{2\sqrt{x^2+4x-4}}dx=\underbrace{\int\frac{2x+4}{2\sqrt{x^2+4x-4}}dx}_{J_1}+\underbrace{\int\frac{2}{2\sqrt{x^2+4x-4}}dx}_{J_2}$$

$$J_1=\int\frac{2x+4}{2\sqrt{x^2+4x-4}}dx \Rightarrow \begin{array}{c} x^2+4x-4=u \\ (2x+4)dx=du \end{array} \Rightarrow J_1=\int\frac{1}{2\sqrt{u}}du=\sqrt{u}\ \ldots (1)$$

$$J_2=\int\frac{1}{\sqrt{x^2+4x-4}}dx=\int\frac{1}{\sqrt{(x+2)^2-8}}dx \Rightarrow \begin{array}{l} (x+2)=2\sqrt{2}\sec t \\ dx=2\sqrt{2}\sec t.\tan t.dt \end{array}$$

$$=\int\frac{1}{2\sqrt{2}\tan t}.2\sqrt{2}\sec t.\tan t.dt=\int\sec t.dt=\ln|\tan t+\sec t|\ \ldots (2)$$

$$J=\sqrt{x^2+4x-4}+\ln\left|\frac{x+2+\sqrt{x^2+4x-4}}{2\sqrt{2}}\right|+c$$

⑥ $f(x)=e^{\frac{x^2+1}{x}}$ için $f(x)=(e^x)\circ\left(\frac{x^2+1}{x}\right)$ old.dan

→ $D_f=\mathbb{R}\setminus\{0\}$ dır.

$f(-x)=e^{\frac{x^2+1}{-x}}$ olup $f(-x)\neq f(x)$ & $f(-x)\neq -f(x)$ old.dan ne tek nede çifttir. $x=0\notin D_f$ olup

$f(x)>0$ old.dan eksenleri kesmez.

→ $\lim\limits_{x\to 0^+} f(x)=+\infty$, $\lim\limits_{x\to 0^-} f(x)=0$ olup sağdan düşey asimtot $x=0$ dır.

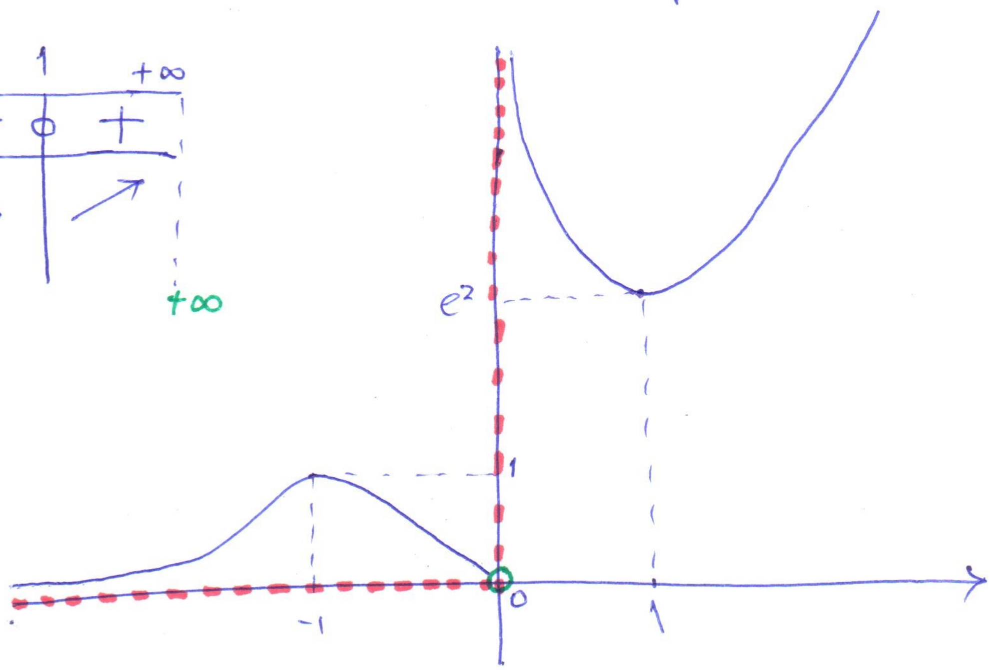
$\lim\limits_{x\to +\infty} f(x)=+\infty$, $\lim\limits_{x\to -\infty} f(x)=0$ olup $y=0$ yatay $(-\infty)$ asimtottur.

→ $f'(x)=e^{\frac{x^2+1}{x}}\cdot\left(\frac{x^2+1}{x}\right)'=e^{\frac{x^2+1}{x}}\cdot\frac{2x.x-x^2-1}{x^2}$

$=e^{\frac{x^2+1}{x}}\cdot\frac{x^2-1}{x^2}$ olur. $f'(x)=0\Rightarrow x=\mp 1$ ve $x=0$

kritik noktalardır. $f(-1)=1$, $f(1)=e^2$ olup

x	$-\infty$		-1		0		1		$+\infty$
f'		$+$	0	$-$	‖	$-$	0	$+$	
f	$y=0$	↗		↘	0 ‖ $+\infty$	↘		↗	$+\infty$



(7) P düzgün ve $s(P)=6$ old. dan $\Delta x_1 = \Delta x_2 = \dots = \Delta x_5 = \frac{2}{5}$

olup $P = \left\{-1, -\frac{3}{5}, -\frac{1}{5}, \frac{1}{5}, \frac{3}{5}, 1\right\}$ dir.

$Ü(f,P) = M_1(f).\Delta x_1 + \dots + M_5(f).\Delta x_5$ için $f'(x)$ incelenirse

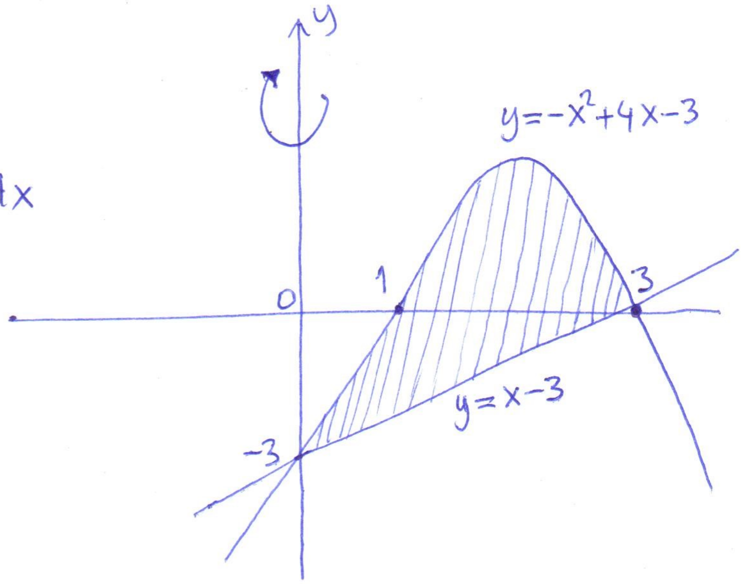
$f'(x) = 3x^2-1$ olup

		$-1/\sqrt{3}$		$1/\sqrt{3}$	
f'	+	0	−	0	+
f	↗		↘		↗

alt aralıklar dikkate alınırsa

$$Ü(f,P) = f\left(-\tfrac{3}{5}\right).\frac{2}{5} + f\left(-\tfrac{1}{\sqrt{3}}\right).\frac{2}{5} + f\left(-\tfrac{1}{5}\right)\frac{2}{5} + \max\left\{f\left(\tfrac{1}{5}\right), f\left(\tfrac{3}{5}\right)\right\}\frac{2}{5}$$
$$+ f(1).\frac{2}{5}$$ bulunur.

(8) $y = -x^2+4x-3 = -(x^2-4x+3) = -(x-1)(x-3)$



Kabuk yöntemiyle

$$V = 2\pi \int_0^3 x\,(-x^2+4x-3-x+3).dx$$
$$= 2\pi \int_0^3 x(-x^2+3x)\,dx$$
$$= 2\pi \left(\frac{-x^4}{4} + x^3\right)\Big|_0^3$$
$$= 2\pi\left(27 - \frac{81}{4}\right) = \frac{2\pi.27}{4} = \frac{27\pi}{2}\ br^3 \text{ olur.}$$